

O Problema dos Peixes Voadores

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Departamento de Matemática Aplicada
Universidade Federal Fluminense

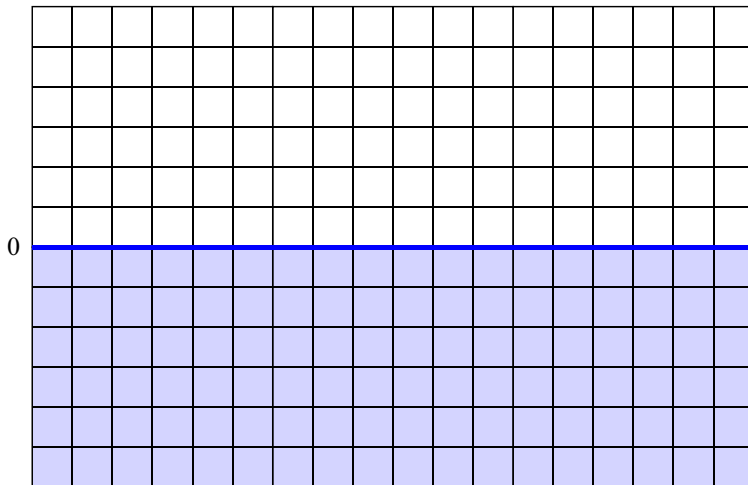
Semana Nacional de Ciência e Tecnologia

5 de outubro de 2007

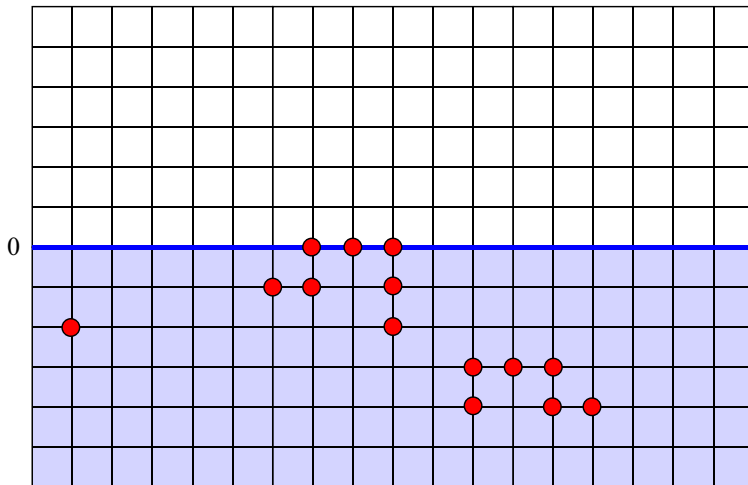
Problema formulado originalmente por
John Conway.

Adaptação do contexto sugerida por
Carlos Tomei.

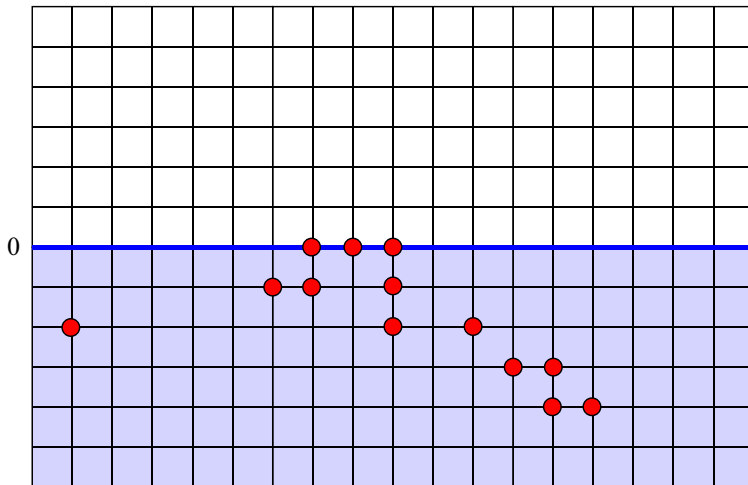
O cenário



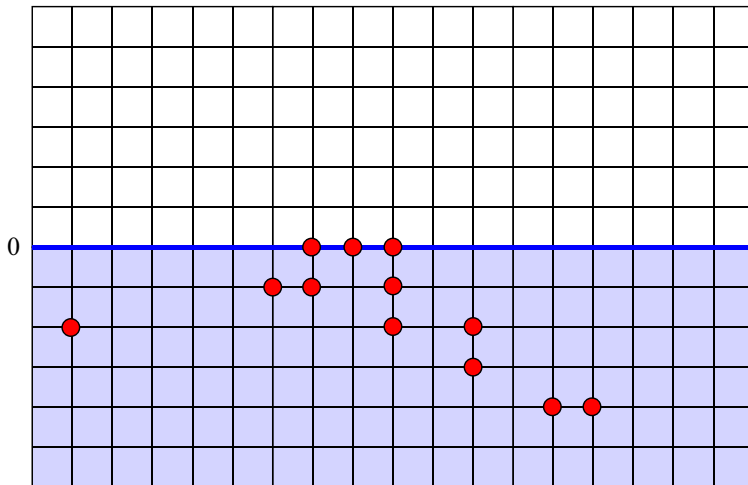
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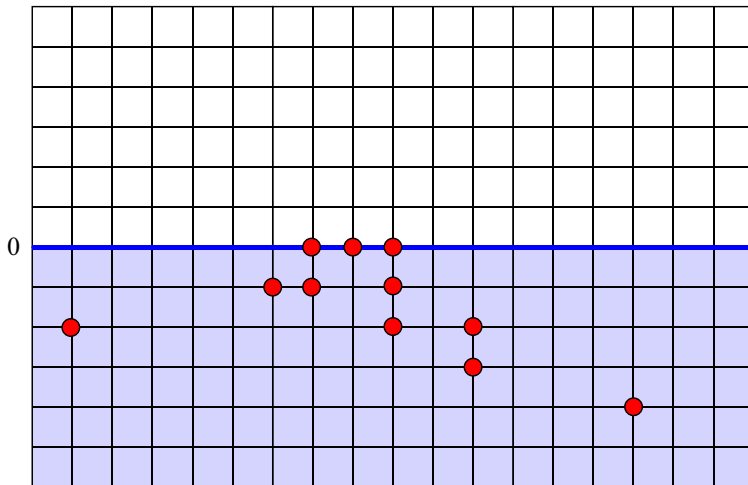
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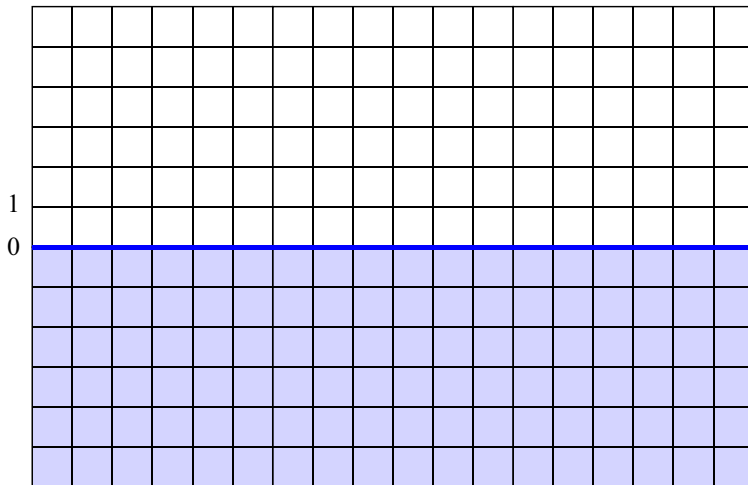
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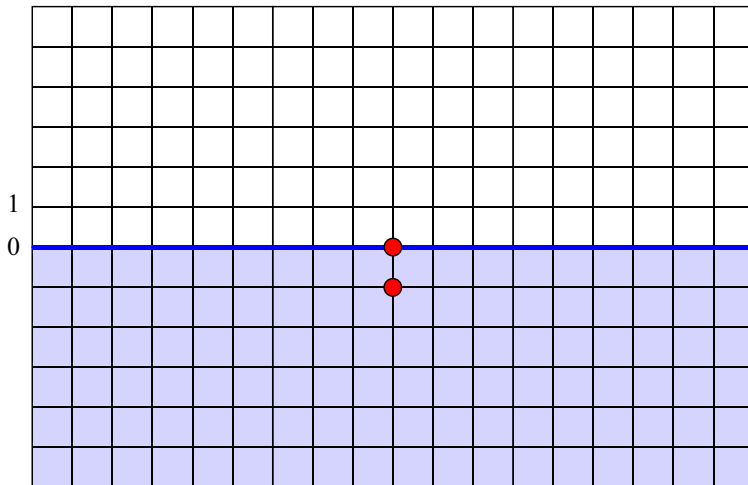
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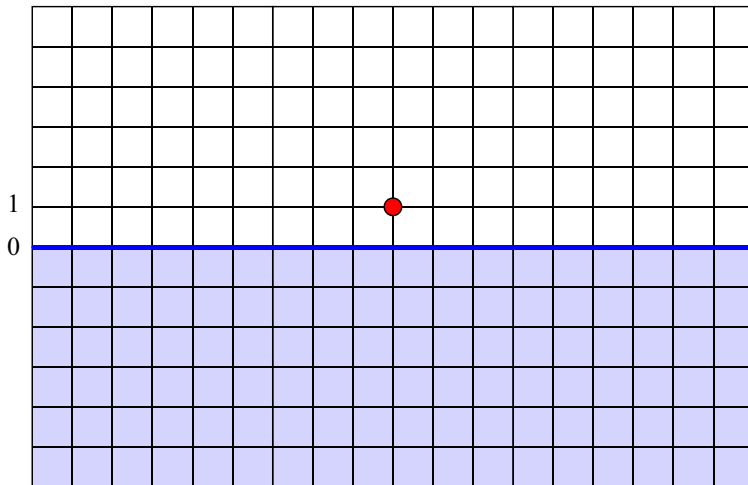
Colocando um peixe no nível 1



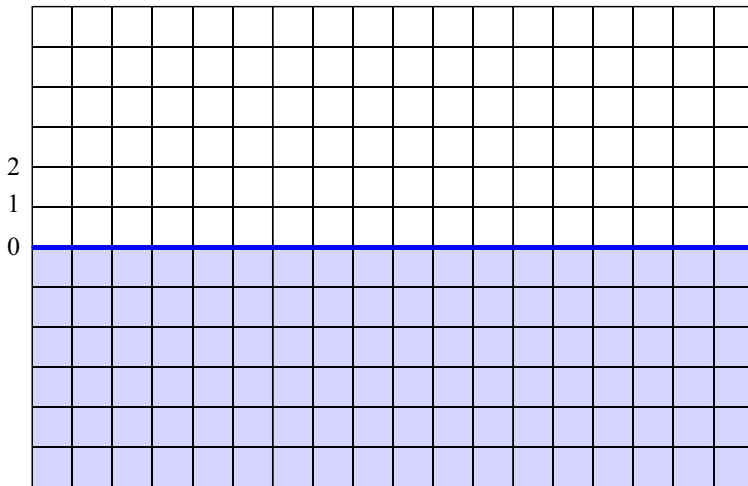
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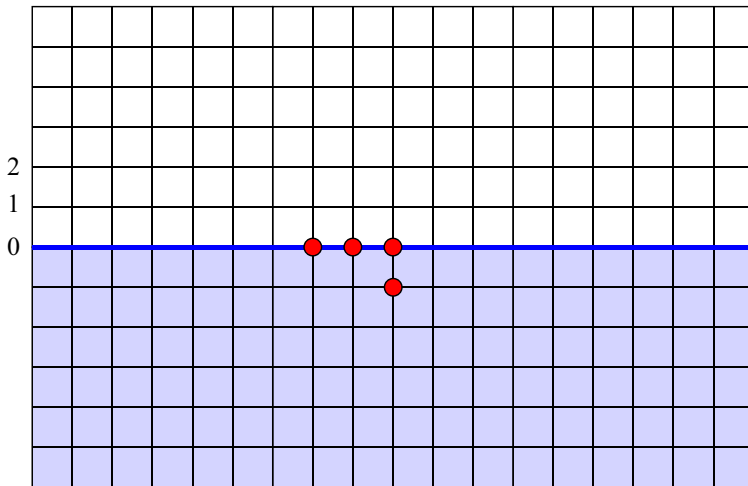
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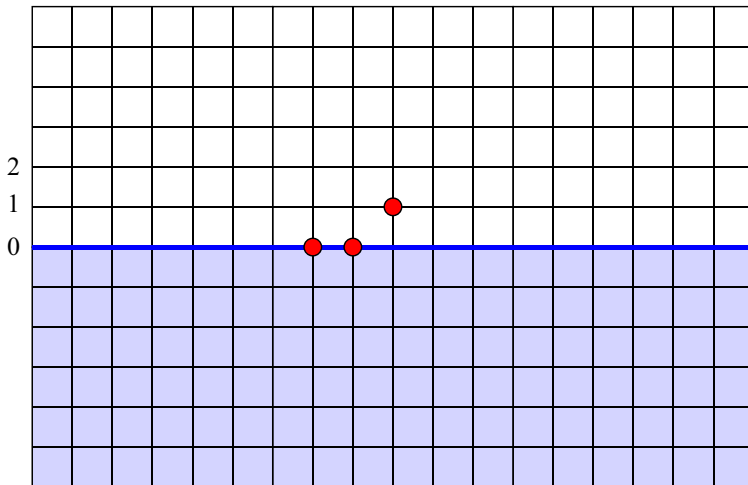
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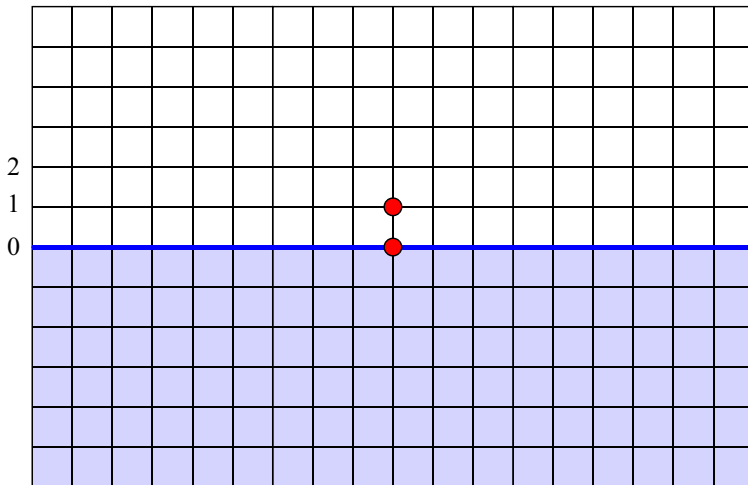
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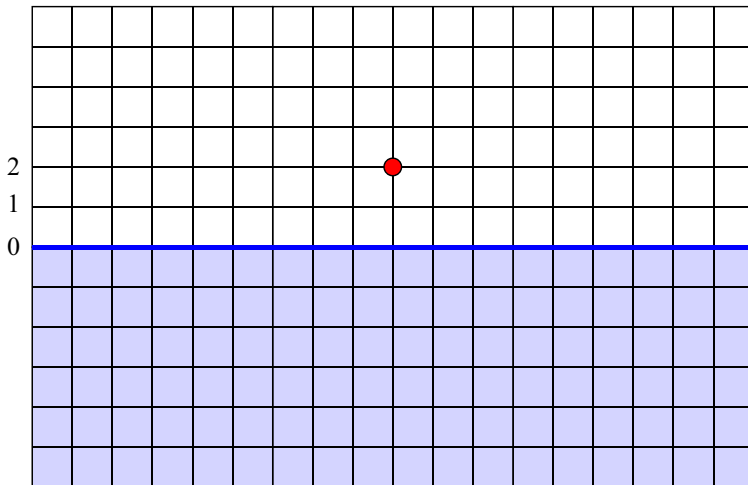
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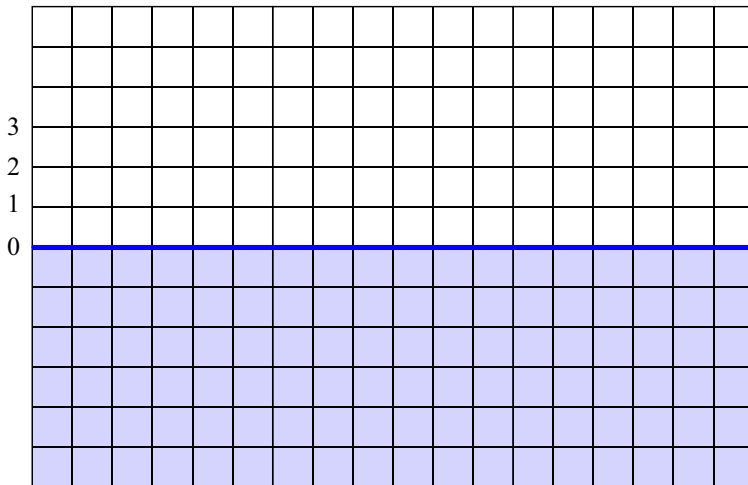
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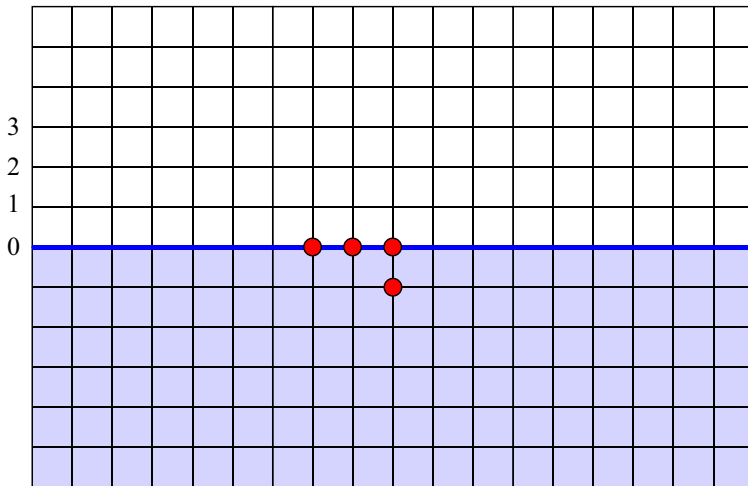
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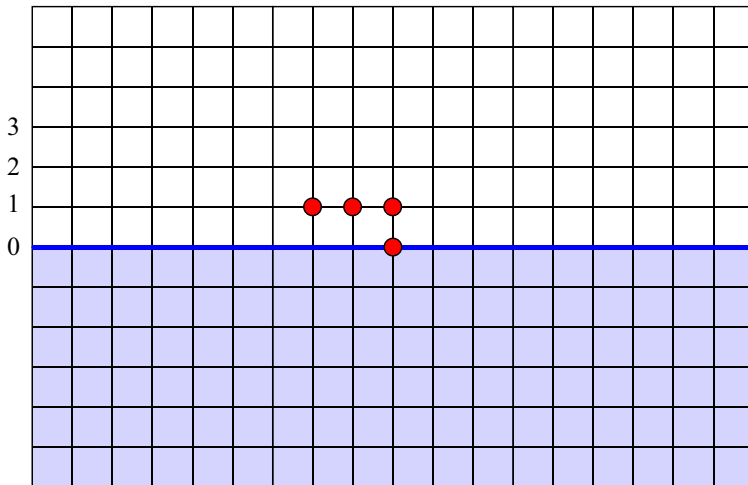
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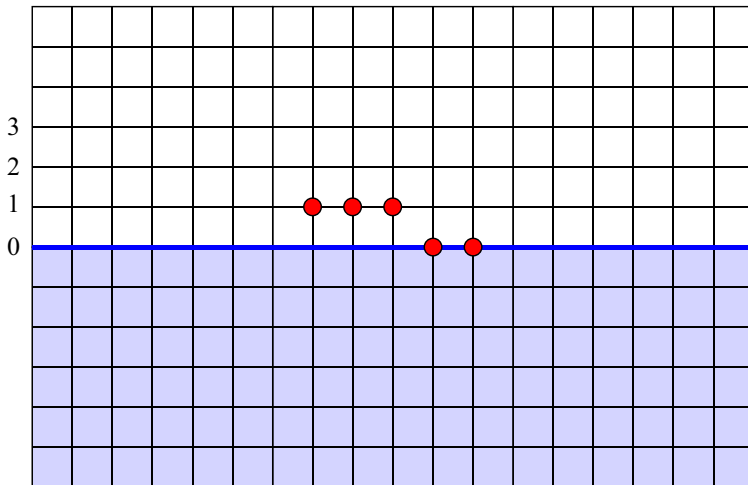
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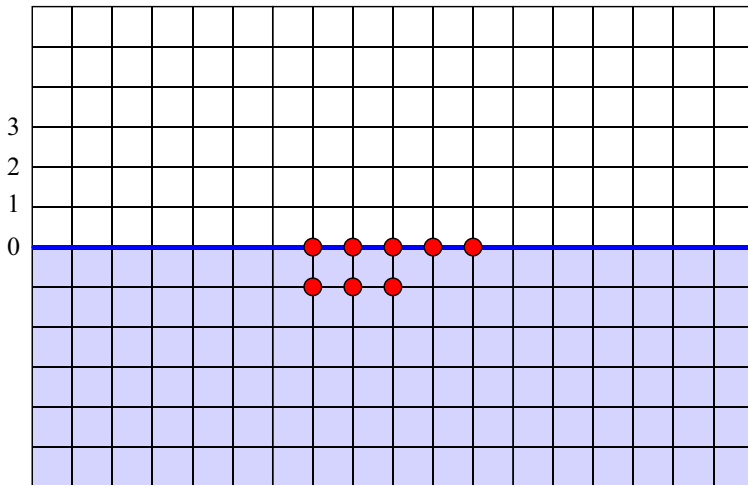
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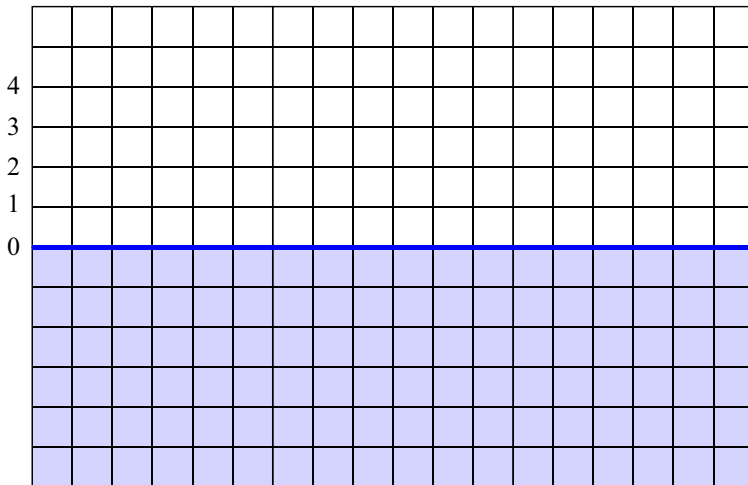
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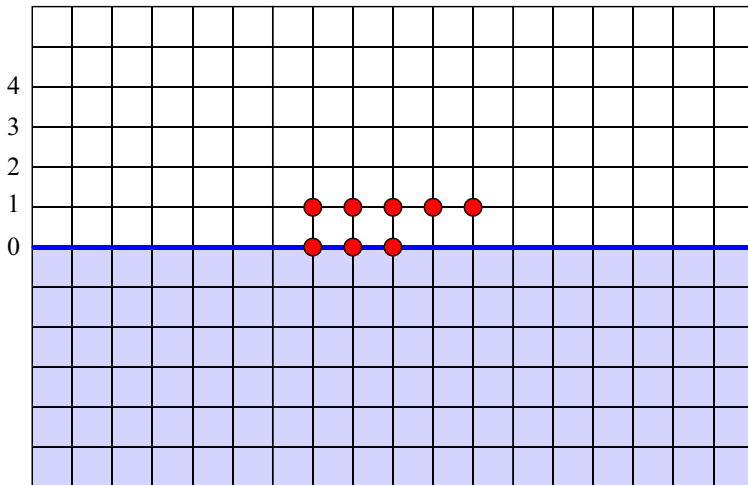
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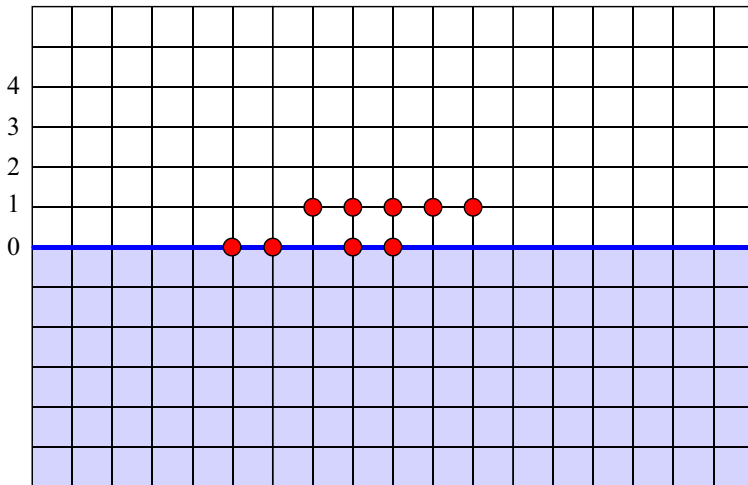
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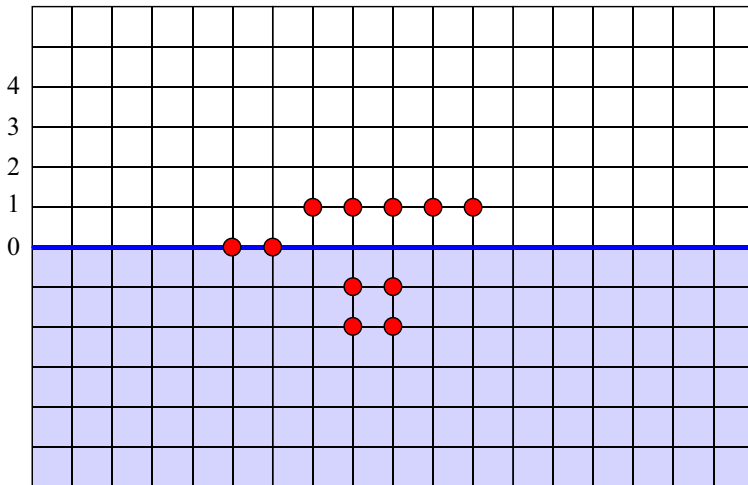
Colocando um peixe no nível 4



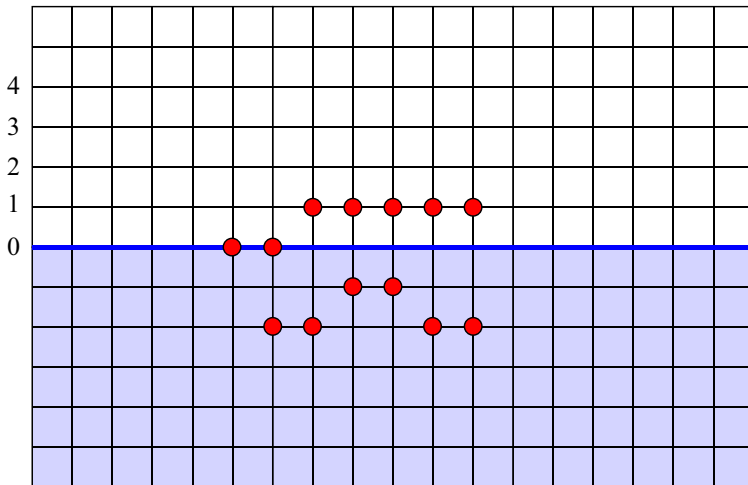
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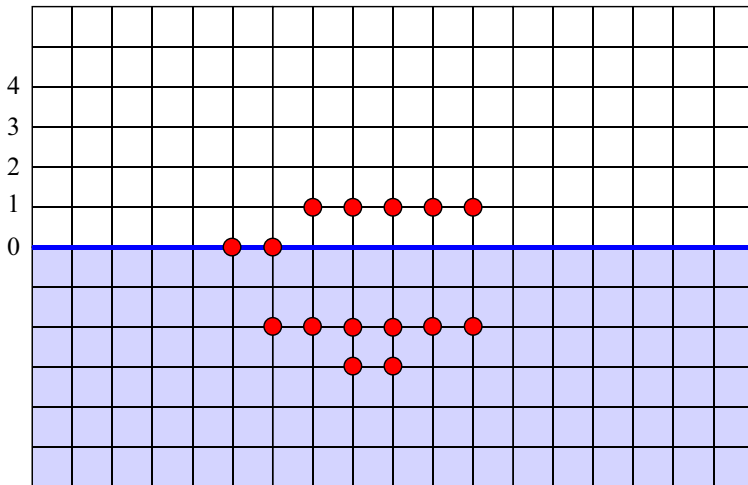
Colocando um peixe no nível 4



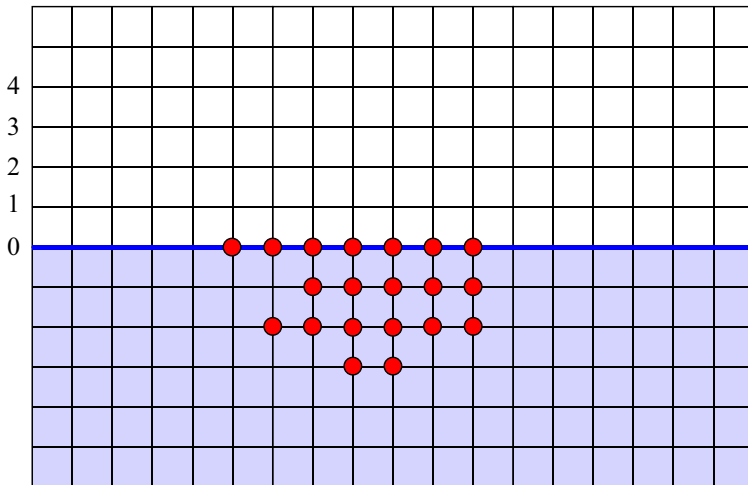
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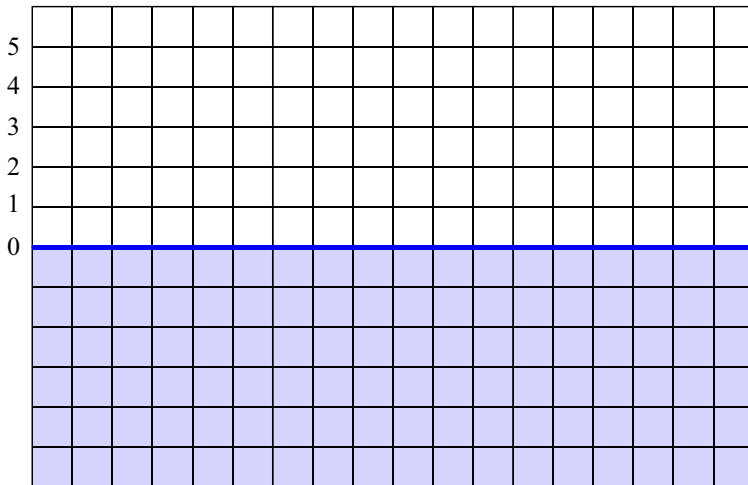
Colocando um peixe no nível 4



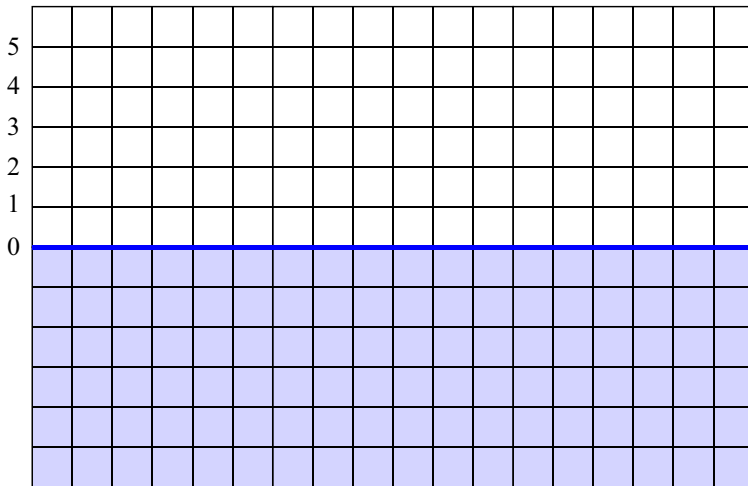
Colocando um peixe no nível 4



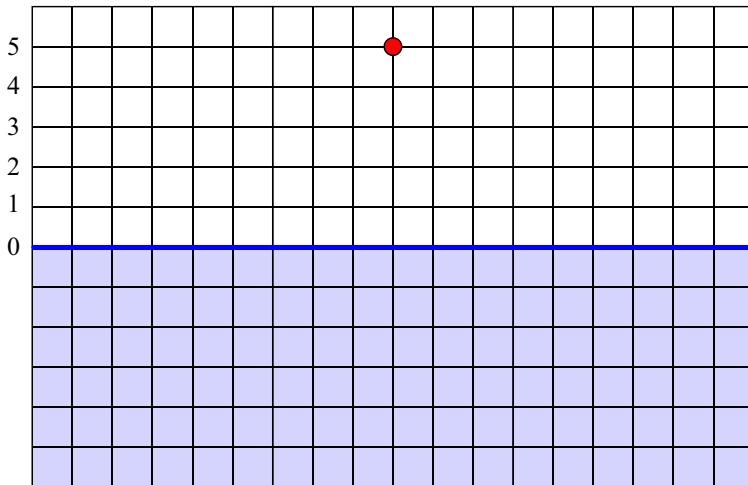
Colocando um peixe no nível 5



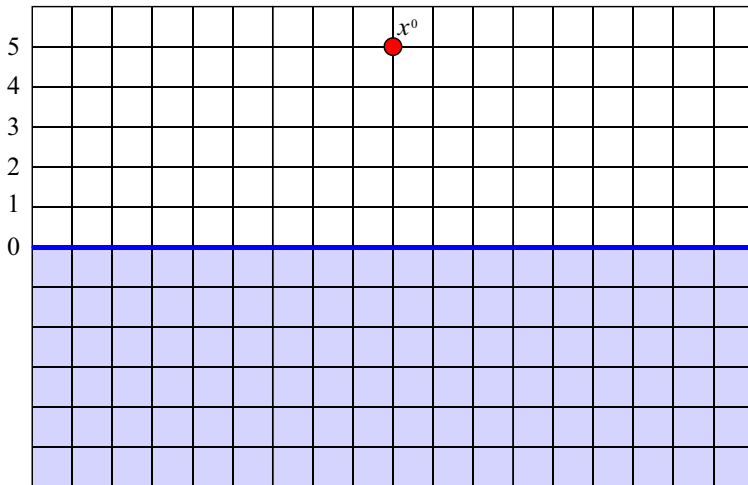
Colocando um peixe no nível 5: não é possível!



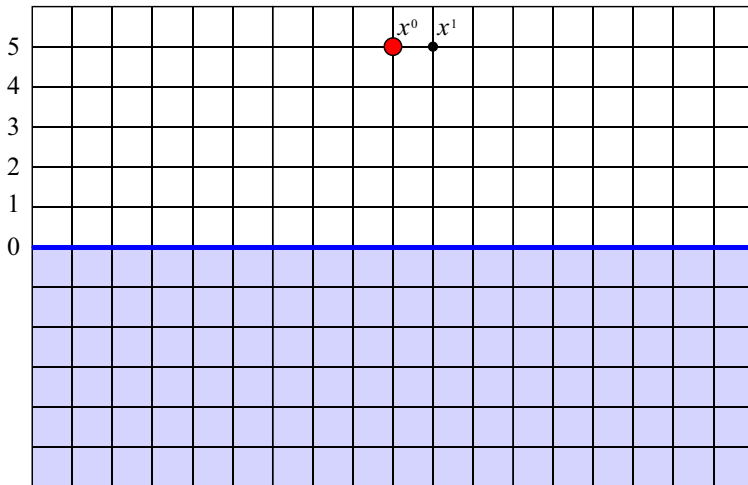
Como provar?



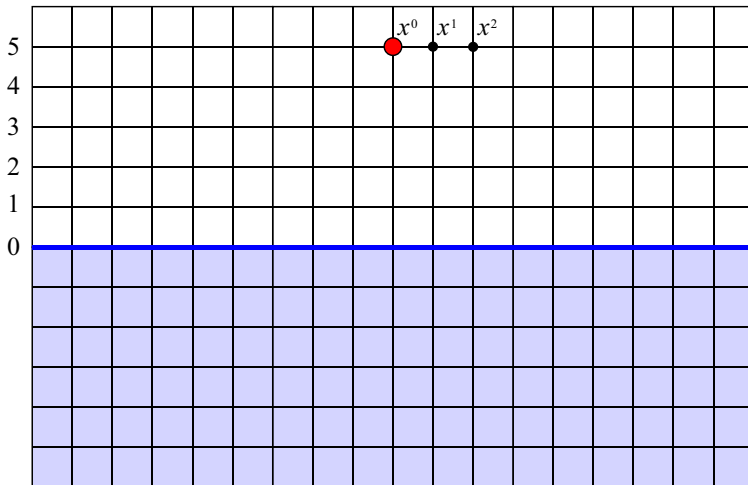
Definindo uma grade de energia



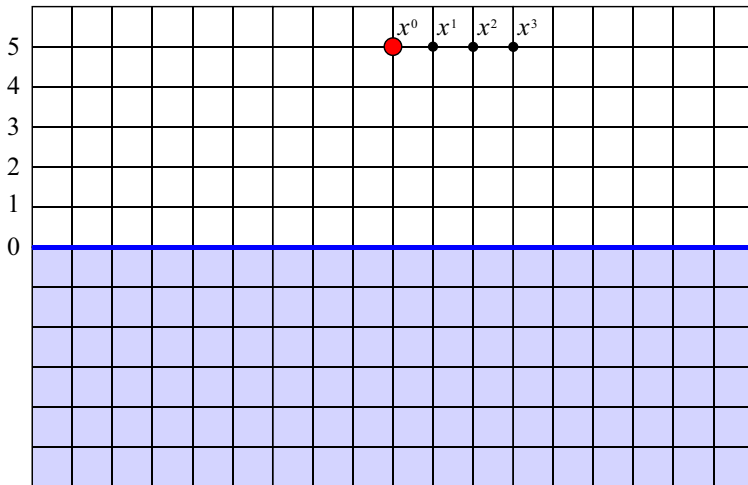
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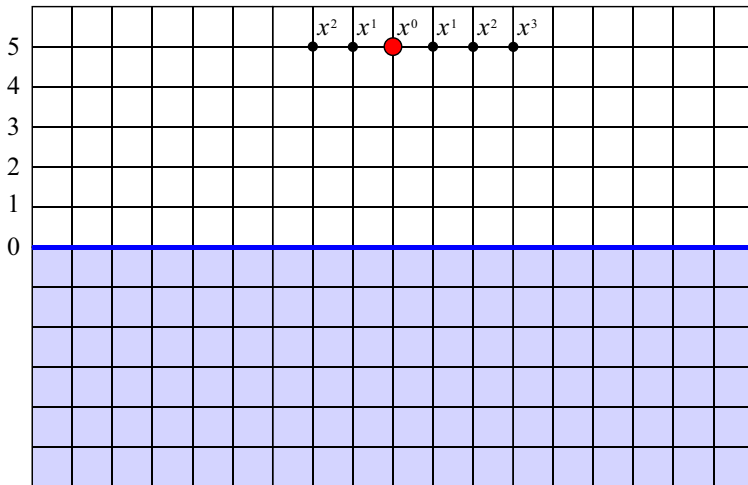
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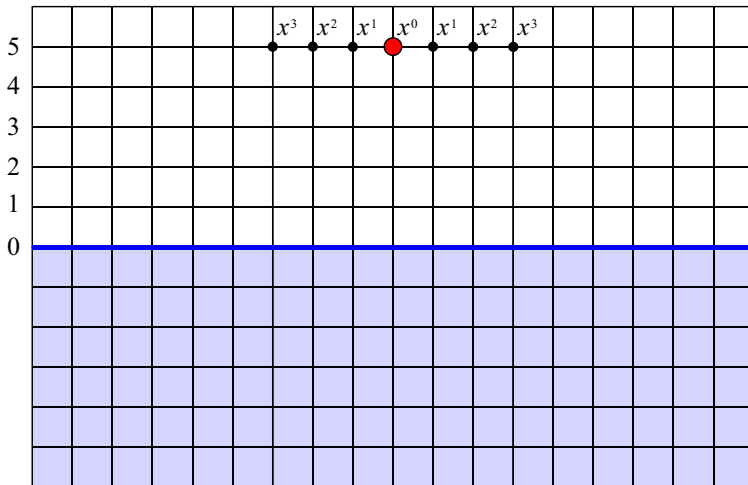
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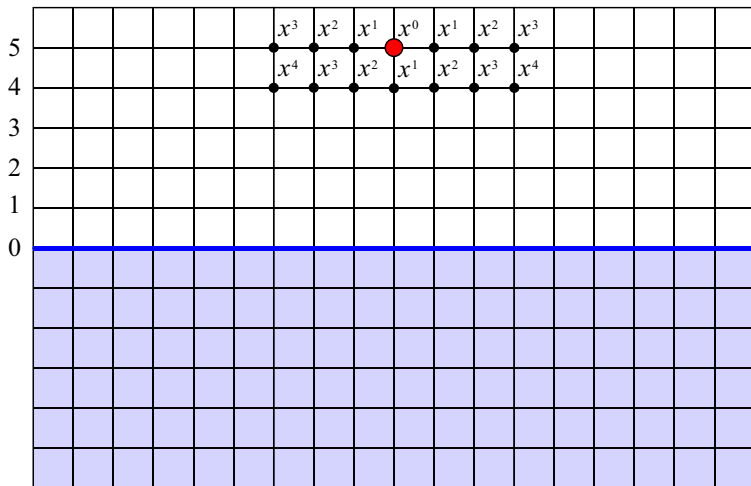
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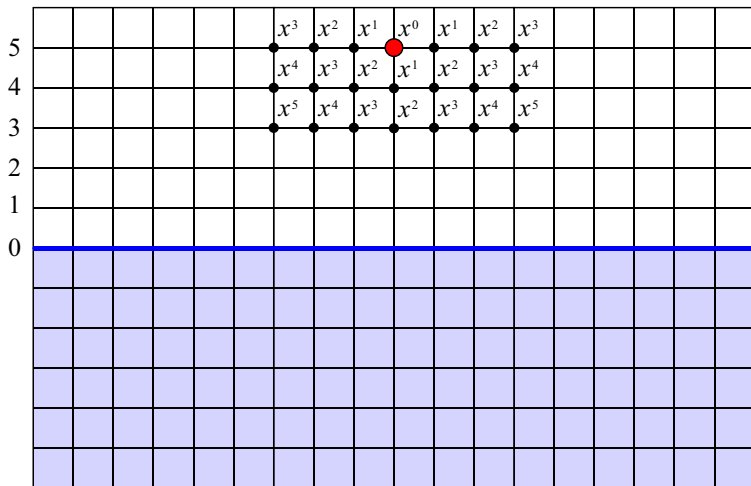
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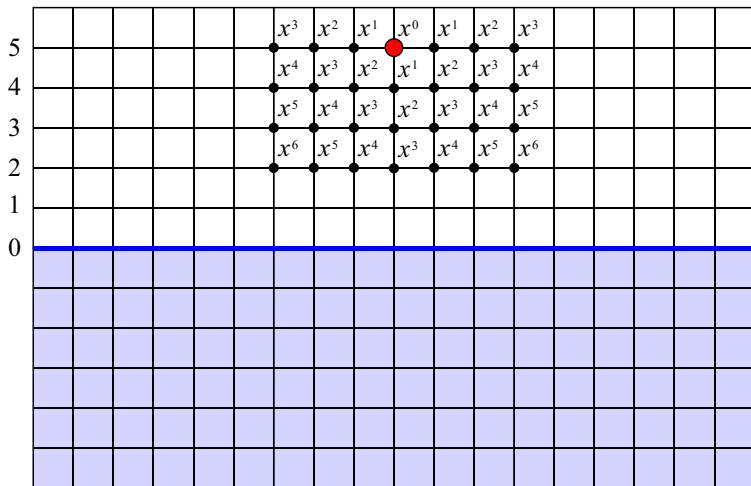
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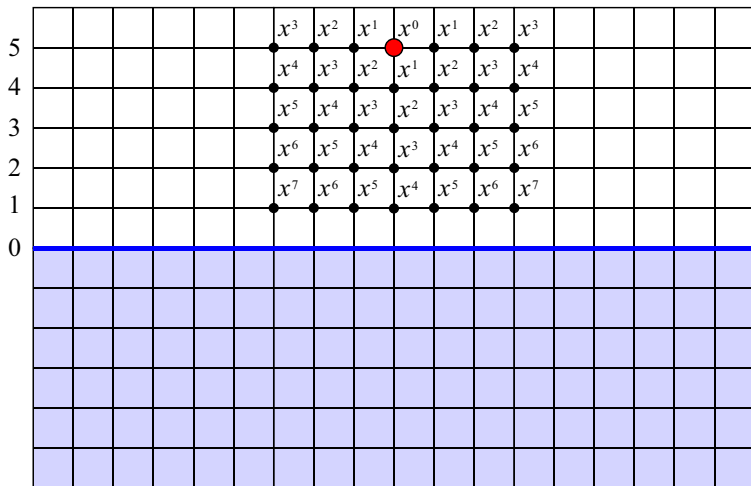
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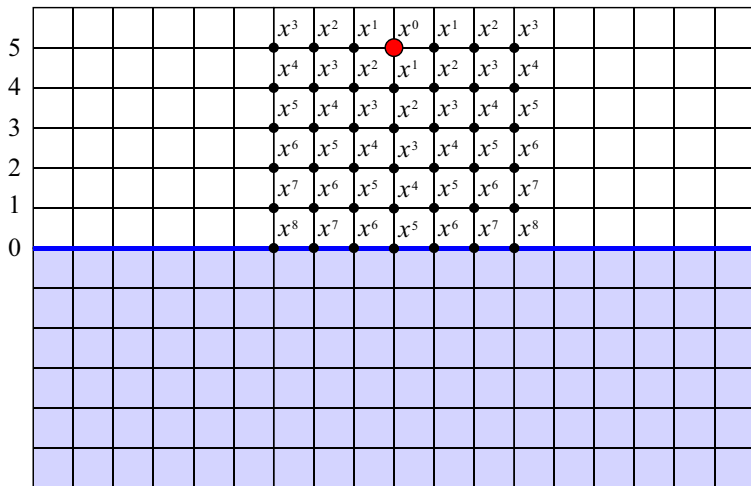
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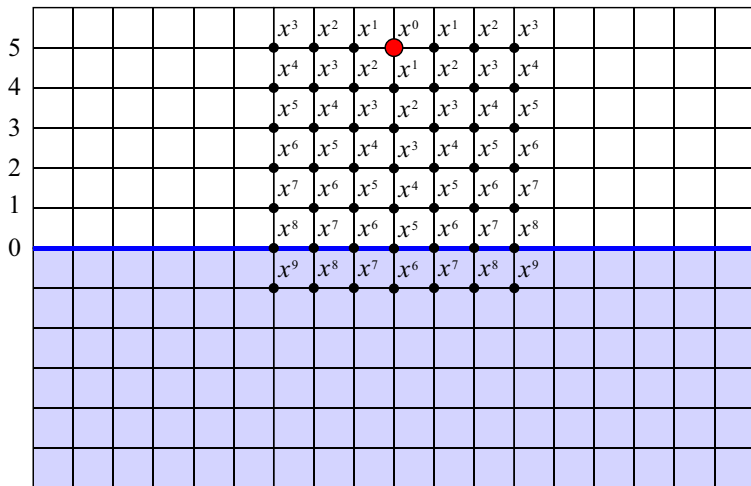
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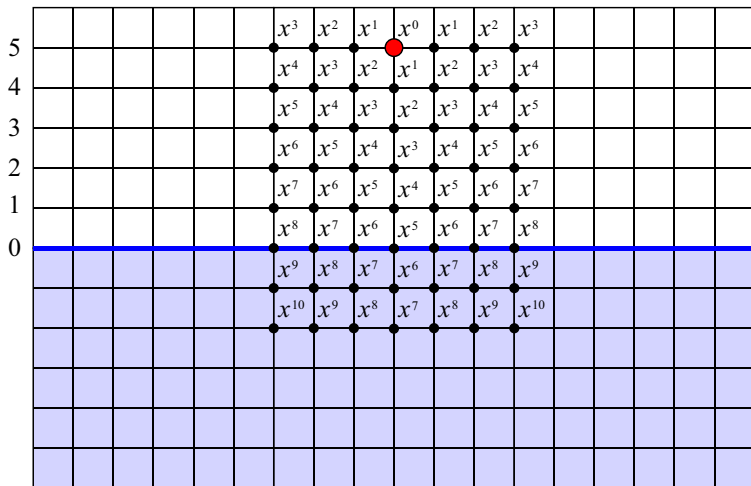
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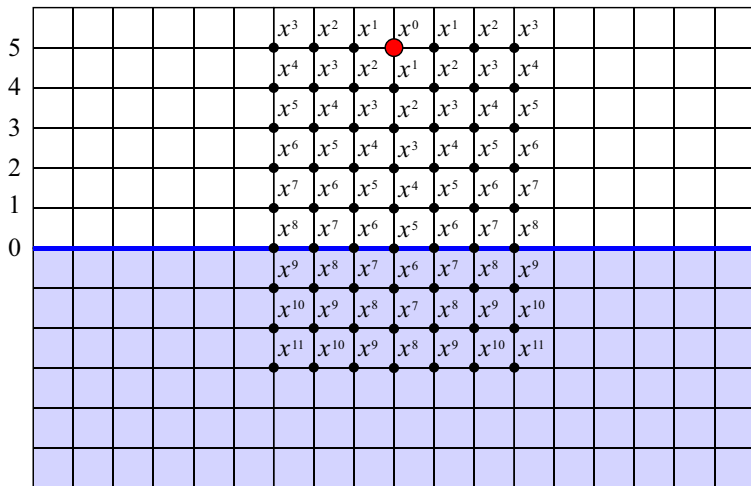
Definindo uma grade de energia



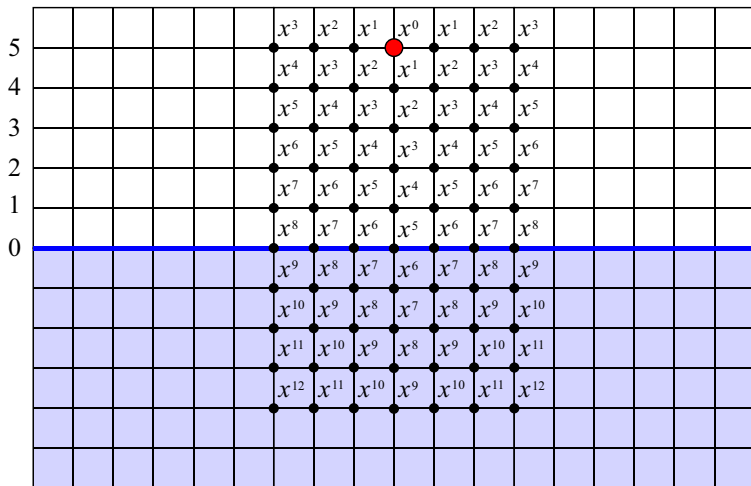
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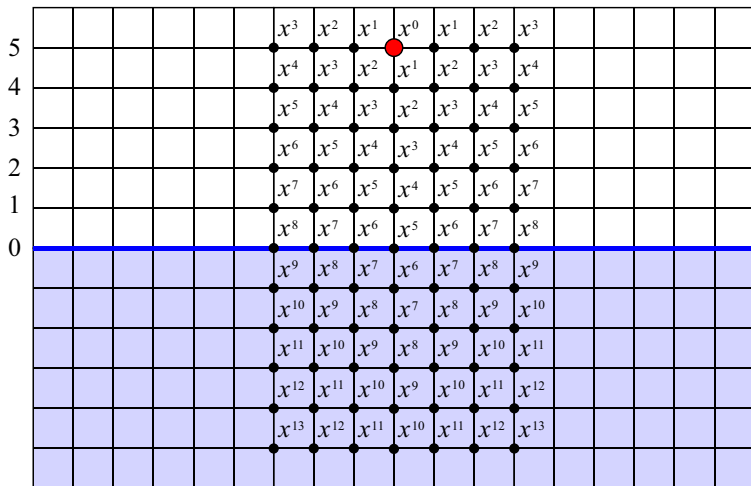
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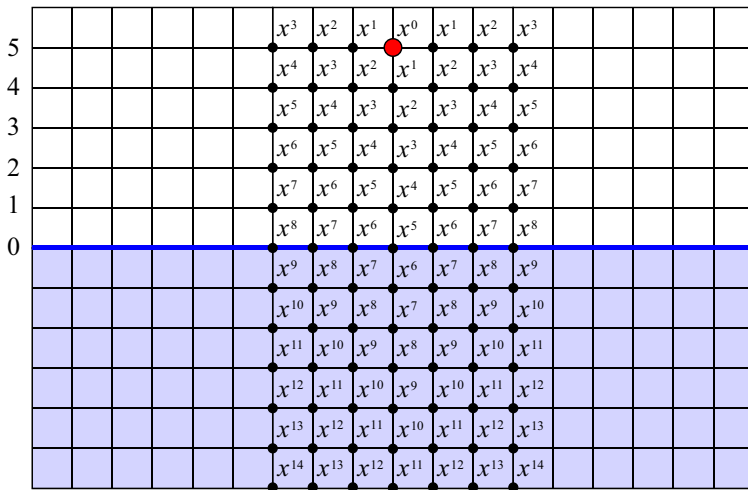
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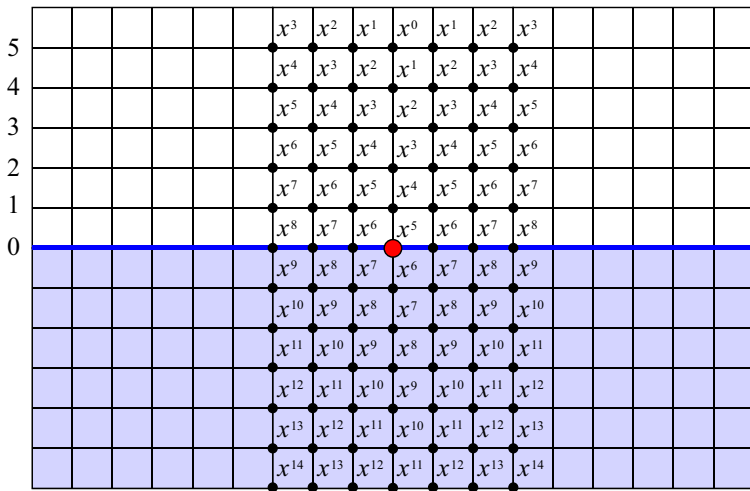
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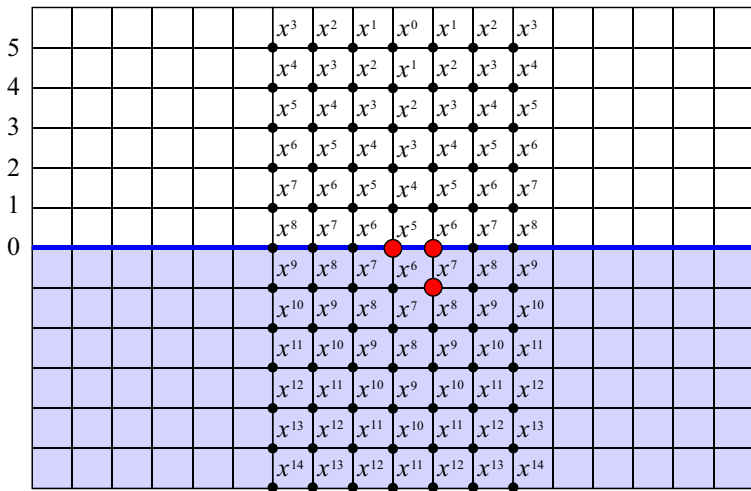
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A energia deste peixe é x^5



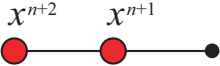
A energia deste cardume é $x^5 + x^6 + x^7$



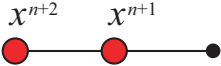

Como devemos escolher o valor de x ?

Antes	Depois	Diferença de Energia
		$x^n - x^{n+1} - x^{n+2}$
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		$x^{n+2} - x^{n+1} - x^n$

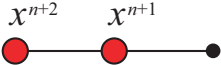

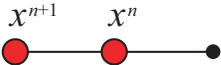
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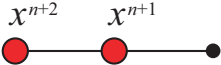

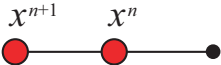

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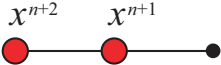

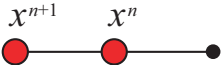

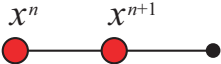
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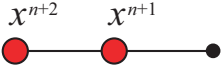

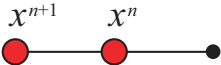

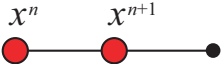

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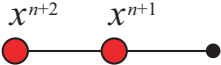

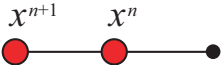

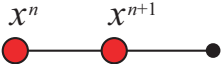

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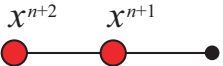

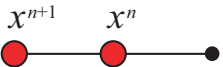

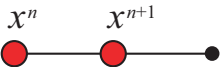

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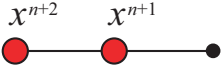

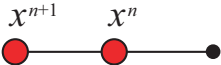

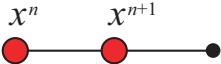

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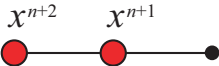

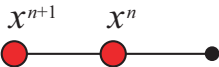

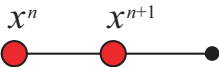

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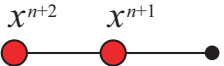

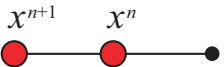

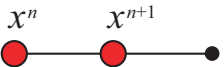

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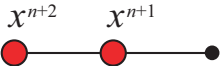

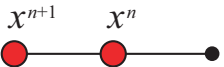

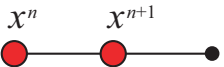

Antes	Depois	Diferença de Energia
		$x^n(1 - x - x^2)$
		$-x^n$
		$x^n(x^2 - x - 1)$

Como devemos escolher o valor de x ?

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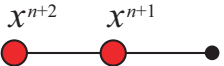

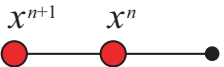

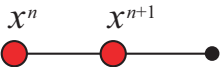

$$1 - x - x^2 = 0$$

Como devemos escolher o valor de x ?

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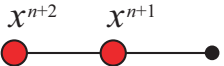

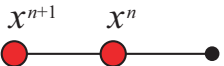

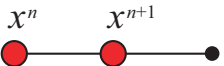

$$1 - x - x^2 = 0 \Rightarrow x = \frac{-1 - \sqrt{5}}{2} \quad \text{ou} \quad x = \frac{-1 + \sqrt{5}}{2}.$$

Como devemos escolher o valor de x ?

Antes	Depois	Diferença de Energia
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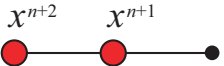

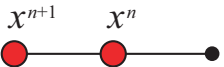

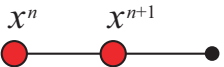

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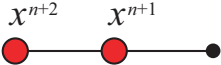

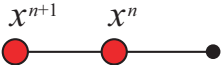

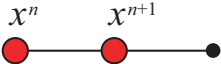

$$x = \frac{-1 + \sqrt{5}}{2} \quad \text{é} \quad > 0 \quad \text{e} \quad < 1.$$

Como devemos escolher o valor de x ?

Antes	Depois	Diferença de Energia
		$x^n(1 - x - x^2)$
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		$x^n(x^2 - x - 1)$

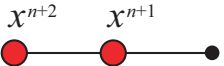

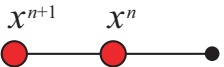

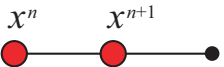

$$x^n(1 - x - x^2) = 0$$

Como devemos escolher o valor de x ?

Antes	Depois	Diferença de Energia
		$x^n(1 - x - x^2)$
		$-x^n$
		$x^n(x^2 - x - 1)$

$$-x^n < 0$$

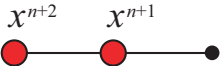

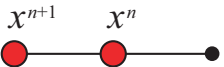

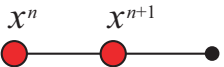

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$$x^n(x^2 - x - 1)$$

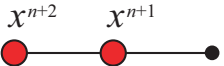

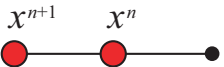

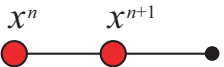

$$< 0$$

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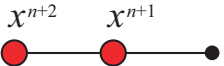

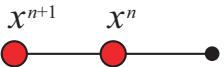

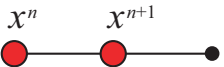

$$x^n(x^2 - x - 1) = x^n(1 - x - x - 1) < 0$$

Como devemos escolher o valor de x ?

Antes	Depois	Diferença de Energia
		$x^n(1 - x - x^2)$
		$-x^n$
		$x^n(x^2 - x - 1)$

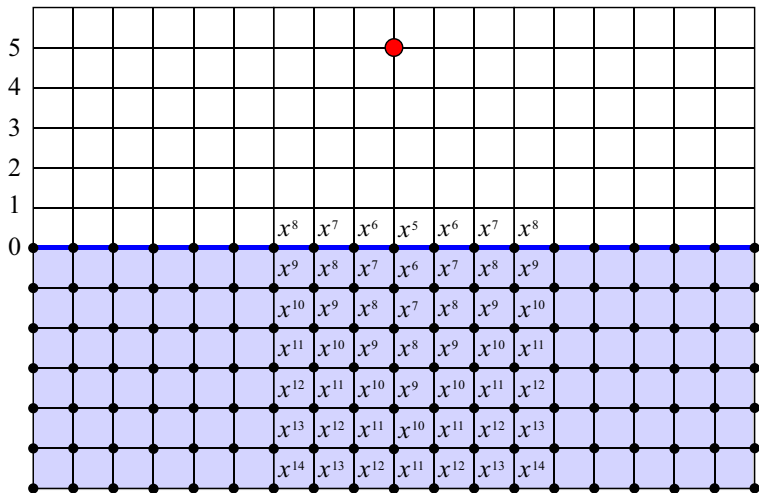
$$x^n(x^2 - x - 1) = x^n(1 - x - x - 1) = -2x^{n+1} < 0$$

Como devemos escolher o valor de x ?

Antes	Depois	Diferença de Energia
 x^{n+2} x^{n+1}	 x^n	$x^n(1 - x - x^2)$
 x^{n+1} x^n	 x^{n+1}	$-x^n$
 x^n x^{n+1}	 x^{n+2}	$x^n(x^2 - x - 1)$

Moral: em cada jogada, a energia de um cardume nunca aumenta!

Qual é a energia de todo o oceano?



Qual é a energia de todo o oceano?

$$\begin{aligned} & x^5 + x^6 + x^7 + \dots \\ & \quad + \\ & 2 \cdot (x^6 + x^7 + x^8 + \dots) \\ & \quad + \\ & 2 \cdot (x^7 + x^8 + x^9 + \dots) \\ & \quad + \\ & \quad \vdots \end{aligned}$$

Qual é a energia de todo o oceano?

$$\begin{aligned} & x^5 \cdot (1 + x + x^2 + \dots) \\ & \quad + \\ & 2x^6 \cdot (1 + x + x^2 + \dots) \\ & \quad + \\ & 2x^7 \cdot (1 + x + x^2 + \dots) \\ & \quad + \\ & \quad \vdots \end{aligned}$$

Qual é a energia de todo o oceano?

$$\begin{aligned} & x^5 \cdot \frac{1}{1-x} \\ & + \\ & 2x^6 \cdot \frac{1}{1-x} \\ & + \\ & 2x^7 \cdot \frac{1}{1-x} \\ & + \\ & \vdots \end{aligned}$$

Qual é a energia de todo o oceano?

$$x^5 \cdot \frac{1}{1-x} + 2x^6 \cdot \frac{1}{1-x} + 2x^7 \cdot \frac{1}{1-x} + \dots$$

||

$$\frac{x^5}{1-x} + \frac{2x^6}{1-x} \cdot (1 + x + x^2 + \dots)$$

||

$$\frac{x^5}{1-x} + \frac{2x^6}{1-x} \cdot \frac{1}{1-x}$$

||

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||

$$x^3 + 2x^2$$

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Qual é a energia de todo o oceano?

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||

$$x \cdot x^2 + 2x^2$$

||

$$x(1 - x) + 2x^2$$

||

$$x - x^2 + 2x^2$$

||

$$x + x^2$$

||

$$x + 1 - x$$

||

$$1$$

Qual é a energia de todo o oceano?

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||

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||

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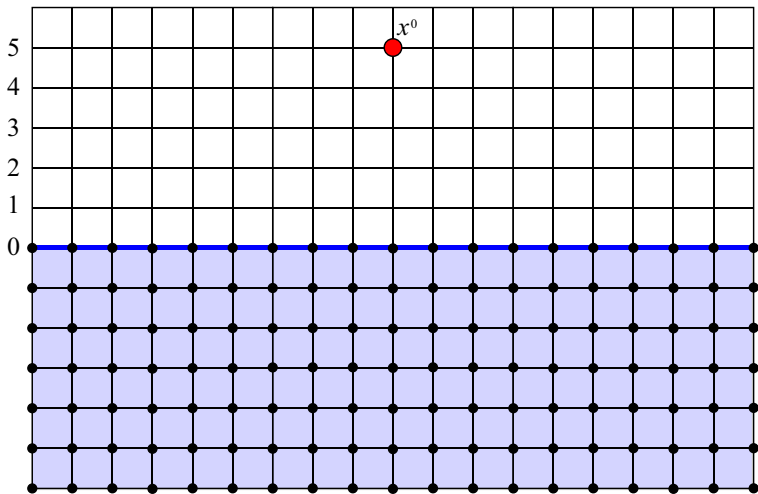
||

$$x + 1 - x$$

||

1

Não é possível colocar um peixe no nível 5!



Referência:

<http://www.cut-the-knot.org/proofs/checker.shtml>

Obrigado!